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NOTE

Two Comments on Filtering (Artificial Viscosity) for Chebyshev and Legendre Spectral and Spectral Element Methods: Preserving Boundary Conditions and Interpretation of the Filter as a Diffusion

A common strategy for reducing numerical noise with a Chebyshev or Legendre spectral method is to filter the coefficients, that is, to replace the truncated Chebyshev series u_N by its filtrate $u_F(N)$,

$$u_N(x) = \sum_{j=0}^{N} a_j T_j(x) \rightarrow u_F(x; N) = \sum_{j=0}^{N} a_j \sigma(j/N) T_j(x) \tag{1}$$

for some filter function $\sigma(\theta)$. The rationale for filtering and choices of good σ are reviewed in [2, 7, 11].

Unfortunately, filtering violates boundary conditions. For example, if $u_N(\pm 1) = 0$, the filtered function is not zero at both endpoints except in special cases. The goal of this note is to propose a simple modification to filtering so that $u_F(x)$ satisfies the same boundary conditions as $u_N(x)$.

The key idea is to rewrite $u_N(x)$ in terms of new basis functions $\phi_j(x)$ which individually satisfy homogeneous boundary conditions and then apply the filter to modify the coefficients b_j of this new expansion. One can multiply the coefficients of these basis functions by arbitrary numbers without disturbing the boundary conditions. The filtered sum can then be converted back into the original Chebyshev or Legendre basis.

There are several technical details. The first is: What if the boundary conditions are *inhomogeneous*? The answer is that one can split the solution $u(x)$ into the sum of a low degree polynomial which satisfies the inhomogeneous boundary conditions plus a sum over the basis functions ϕ_j that satisfy the equivalent homogeneous boundary conditions.

To explain the idea, it is sufficient to specialize to Dirichlet boundary conditions:

$$u(-1) = \alpha; \qquad u(1) = \beta. \tag{2}$$

If the basis functions are chosen so that they satisfy homogeneous Dirichlet boundary

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conditions, that is,

$$\phi_j(\pm 1) = 0 \tag{3}$$

then without approximation one may write

$$u_N(x) = (\alpha + \beta)/2 + (\beta - \alpha)x/2 + \sum_{j=2}^{N} b_j \phi_j(x). \tag{4}$$

It is easy to construct low degree polynomials which fit the inhomogeneous boundary conditions—for very general boundary conditions—as described in the author's book [1, 3]. In more than one space dimension, one can use "transfinite interpolation" [3, 6, 12].

The second technical detail is: Which basis should one use? There is an infinite number of linear combinations of Chebyshev polynomials that satisfy each set of boundary conditions. For homogeneous Dirichlet conditions such as (3), one such basis is

$$\phi_{2j}(x) \equiv T_{2j}(x) - T_0; \qquad \phi_{2j+1}(x) \equiv T_{2j+1}(x) - T_1. \tag{5}$$

However, if we filtered the coefficients in this basis, any changes in high degree Chebyshev coefficients would also cause changes in the very lowest Chebyshev coefficients (those of T_0 and T_1). This is very unsatisfactory for filtering.

Heinrichs [8–10] has shown that the basis

$$\phi_j(x) \equiv (1 - x^2)T_j(x) = \frac{1}{4}\{2T_j(x) - T_{|j-2|}(x) - T_{j+2}(x)\} \tag{6}$$

greatly reduces roundoff error because the maximum values of its second derivative at and near the endpoints is only $O(j^2)$ versus $O(j^4)$ for the first basis or for the Chebyshev polynomials themselves. However, this basis has the modest defect that each element is a weighted sum of *three* Chebyshev polynomials, as can be shown by using the recurrence relation for the Chebyshev polynomials, so it is more expensive to convert to and from an ordinary Chebyshev series.

The third basis, and the one we shall analyze, is

$$\phi_j(x) \equiv T_{j+2}(x) - T_j(x). \tag{7}$$

Let $\overline{b}_j$ and $\overline{a}_j$ denote the filtered coefficients in the ϕ and Chebyshev series, respectively, and let $\sigma_j = \sigma(j/N)$. Then, given only the unfiltered Chebyshev coefficients a_j, both the unfiltered ϕ coefficients and the filtered Chebyshev coefficients can be computed in a single loop. The even and odd degree coefficients are uncoupled and may be computed either separately or together. For simplicity, we give only a single set of formulas which apply to either even or odd where N is to be interpreted as the degree of the largest coefficient of the appropriate parity.

The initialization is

$$\lambda = \sigma_{N-2}\, a_N, \qquad b_{N-2} = a_N, \qquad \overline{a}_N = \lambda, \qquad \rho = \lambda \tag{8}$$

and the loop, which is executed with index j starting from $(N-2)$ in steps of (-2) until j is the smallest integer of the appropriate parity which is larger than 1:

$$\begin{aligned} b_{j-2} &= b_j + a_j, \qquad j = (N-2), (N-4), \ldots, j_{min} > 1 \\ \lambda &= \sigma_{j-2}\, b_{j-2} \\ \overline{a}_j &= \lambda - \rho \\ \rho &= \lambda. \end{aligned} \tag{9}$$

Note that the two lowest Chebyshev coefficients, a_0 and a_1, are not changed by the filtering.

Similar basis functions for more general boundary conditions can be invented using the exact formula for the endpoint derivatives of the Chebyshev polynomials:

$$\frac{d^p T_n}{dx^p}(\pm 1) = (\pm 1)^{n+p} \prod_{k=0}^{p-1} \frac{n^2 - k^2}{2k+1}. \tag{10}$$

Thus, a basis satisfying homogeneous Neuman boundary conditions is

$$\phi_j = \frac{j^2}{(j+2)^2} T_{j+2}(x) - T_j(x), \qquad j = 0, 1, \ldots. \tag{11}$$

Again, one can calculate the filtered from the unfiltered coefficients in a single loop.

The Legendre polynomials, like the Chebyshev, are normalized so that $P_j(\pm 1) = (\pm 1)^j$. Thus, a good basis that satisfies homogeneous Dirichlet conditions is

$$\phi_j(x) = P_{j+2}(x) - P_j(x) \tag{12}$$

and the Chebyshev formula for computing the filtered coefficients, (8) plus (9), applies without modification.

Our second comment is that there is an underlying philosophical question which has been glossed over above: How many boundary conditions are needed when filtering is imposed? The reason that this is a non-trivial issue is that filtering can be interpreted as adding a damping term to the original equations of motion.

Even though the filtering may be performed as a separate procedure between time steps, without benefit of the usual time-integration schemes, sequential operations are equivalent to simultaneous operations if the time step is sufficiently small. That is to say, if $u_t = L(u)$ is the original undamped equation or system of equations and if the operator V is chosen so that the effect of the filter is equivalent to integrating the linear problem $u_t = Vu$ over an interval of one time step τ, then the solution of

$$\begin{aligned} u_t &= Lu, \qquad t \in [0, \tau] \\ v_t &= Vv, \qquad t \in [0, \tau], v(0) = u(\tau) \end{aligned} \tag{13}$$

is equivalent to the solution of the damped system

$$w_t = Lw + Vw, \qquad t \in [0, \tau], w(0) = u(0) \tag{14}$$

in the sense that

$$w(\tau) = v(\tau) + O(\tau^2). \tag{15}$$

For example, in the Fourier basis $\{\exp(ijx)\}$, the filter $\sigma_j = 1 - \nu\tau j^6$ is equivalent to adding the damping term

$$V \equiv \nu u_{xxxxxx}. \tag{16}$$

It is because of this equivalence that "filtering-every-time-step" and "artificial viscosity" are often used loosely as synonyms even though there is a difference, both to the programmer and the mathematician, between adding a damping term to the equations of motion or coding the dissipation as a separate subroutine that is applied between time steps.

Fourier series are normally used only with periodic boundary conditions, so the fact that the sixth derivative has raised the order of the system of partial differential equations does not cause difficulties for the trigonometric basis. However, similar artificial damping terms are often used with non-trigonometric basis sets, too. If the damping raises the order of the differential equations, how can we impose *only* the original Dirichlet boundary conditions and still obtain a well-posed numerical problem?

A partial answer is that if we apply the filter analogous to a sixth-order hyperviscosity to a Legendre basis, the differential operator is

$$V \equiv \mu\{[(1-x^2)u_x]_x\}^3 \leftrightarrow \sigma_j = 1 - \tau\nu[j(j+1)]^3 \tag{17}$$

since the operator V is just the cube of the eigenoperator for Legendre polynomials:

$$[(1-x^2)P_{j,x}]_x = -j(j+1)P_j. \tag{18}$$

The Legendre operator has the form of a viscosity operator, i.e., $[\mu(x)u_x]_x$, but with a viscosity $\mu(x) = 1 - x^2$ which goes to zero at the boundaries!

This seems very weird and counterintuitive, but the spatial variation of the viscosity actually solves two important numerical difficulties. The first is that the more obvious choice of a damping which is proportional simply to the second derivative operator or to higher powers of the second derivative operator gives a Chebyshev or Legendre discretization matrix which is very poorly conditioned; with a truncation at N polynomials, the largest eigenvalue is $O(N^4)$ for the second derivative, $O(N^8)$ for the fourth derivative, etc., [1, 3–5]. This not only amplifies the roundoff error, but requires an extremely short time step unless the damping constant ν is very small, in which case the lower Chebyshev or Legendre polynomials are not damped or filtered at all. In contrast, the powers of the Legendre operator have maximum eigenvalues which are roughly the square root of the derivative matrix of the same order—$O(N^6)$ for the sixth order Legendre damping versus $O(N^{12})$ for a sixth derivative dissipation.

The second reward for using the Legendre eigenoperator is that it is *singular* at the endpoints. This would hardly seem to be a virtue, but because the operator is singular, numerical boundary conditions are not needed, regardless of how many times the Legendre operator is squared or cubed. Instead, the proper boundary conditions are merely that the solution should be analytic at the endpoints in spite of the singularities in the differential equation. The sum of N Legendre polynomials automatically satisfies these behavioral boundary conditions, so no explicit constraints are needed.

Thus, when an artificial damping in the form of the cube of the Legendre eigenoperator is added to a problem whose inviscid form is second order and needs only Dirichlet boundary conditions, it is not necessary to add two additional boundary conditions at each endpoint for mathematical well-posedness. Rather, one must *drop* the Dirichlet boundary conditions of the inviscid conditions. Indeed, this is the whole point of the first half of this note: if one applies a filter to a Chebyshev or Legendre basis without taking special steps to enforce the boundary conditions, then the artificial viscosity will alter the boundary values of the solutions at each time step.

The modified, boundary-preserving filtering described above can be probably expressed in terms of powers or a series of powers of a differential operator which is non-singular at the boundaries and has as its eigenfunctions $\phi_j = P_{j+2} - P_j$ where the $P_j(x)$ are the Legendre polynomials. Unfortunately, we have not been sufficiently clever to find a closed form for such an eigenoperator and perhaps no simple form exists.

Nevertheless, the loss of all boundary conditions when the damping is expressed in terms of the Legendre eigenproblem (without the recursion above for preserving boundary conditions) shows that additional boundary conditions are not necessarily needed when adding an artificial viscosity or filtering.

Similar remarks apply with Chebyshev polynomials. The Chebyshev eigenoperator

$$\sqrt{1-x^2}\,[\sqrt{1-x^2}T_{n,x}]_x = -n^2 T_n \tag{19}$$

is not quite in the form of a fluid mechanical viscosity with a spatially varying viscosity coefficient. Nevertheless, the same mathematical and numerical arguments apply.

It still seems peculiar from a physical viewpoint that the “good” numerical dissipation is one that goes to zero at the endpoints, precisely where all right-thinking fluid dynamicists expect boundary layers. This, however, is precisely the point: if the artificial damping did not vanish at the endpoints, it would introduce spurious boundary layers and require additional, unphysical boundary conditions. If the physics requires a boundary layer, we should add a second derivative with a physically determined viscosity coefficient to provide it, and let the sixth order, artificial viscosity go to zero at the endpoints to leave the physical boundary layer untouched.

The overall conclusion is that if filtering is done properly, we can add an artificial viscosity to a computation using Chebyshev or Legendre polynomials and still preserve the original boundary conditions. Nothing more complicated than a single DO loop is required; the filter coefficients σ_j can be anything we wish; no spurious extra boundary conditions are necessary.

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